

New Derivation of Off-Shell Representations of the Multi-dimensional Affine and Virasoro Algebras

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Abstract

Algebras of currents and diffeomorphisms in arbitrary dimension have extensions which generalize the affine and Virasoro algebras on the circle. A large class of off-shell representations was discovered in Comm. Math. Phys. **214** (2000) 469–491. That paper is not so accessible due to a slightly non-standard normal ordering formalism and cumbersome p -jet calculations. The purpose of the present paper is to simplify the derivation using standard OPE methods and a more field-like formalism.

1 Introduction

The algebra of diffeomorphisms on the circle has a non-trivial central extension, the Virasoro algebra. A natural question is whether this extension can be generalized to diffeomorphisms on higher-dimensional manifolds, in particular on the d -dimensional torus. The answer is affirmative and the multi-dimensional Virasoro algebra $Vir(d)$ was discovered during the 1990s [6, 10], together with many representations [2, 3, 7, 10].

$Vir(d)$ for $d > 1$ differs in many respects from the ordinary Virasoro algebra $Vir(1)$. E.g., the Virasoro-like extension is not central unless $d = 1$. In d dimensions, it is an extension by the module of closed $(d - 1)$ -forms. When $d = 1$, a closed zero-form is a constant function, and the extension is central. In higher dimensions, the Lie algebra extension is still well-defined, but it does not commute with diffeomorphisms.

The off-shell representations are also essentially different. Fock representations of $Vir(1)$ are obtained by quantizing fields on the circle. However, this procedure does not generalize to higher dimensions, because normal ordering gives rise to non-removable infinities. In some sense, this reflects the fact that Quantum Field Theory is incompatible with general-covariant theories like gravity. To construct off-shell representations of $Vir(d)$ with $d > 1$, we must replace the fields with histories in the corresponding spaces of p -jets prior to quantization. After this step is taken, we have a classical representation acting on finitely many functions of a single variable, which can be quantized without the appearance of infinities.

Unfortunately, working with p -jets is quite cumbersome. The purpose of the present paper is to simplify the calculations and to emphasize the close relation to field theory. Locally, a p -jet is simply a Taylor expansion truncated at order p ; in one dimension, the p -jet corresponding the field $\phi(x)$ is

$$\phi(x)|_p = \sum_{m=0}^p \frac{1}{m!} \phi_m(x-q)^m, \quad (1.1)$$

where the Taylor coefficients $\phi_m = d^m \phi / dx^m(q)$.

A Taylor series does not only depend on the function being expanded, but also on the choice of expansion point q , to be identified with the *observer's position*. This means that (1.1) does not commute with the observer's momentum, i.e. the canonical conjugate of q . To obtain an orthogonal set of canonical variables, we replace the *absolute field* $\phi(x) \equiv \phi_A(x)$ with the corresponding *relative field* $\phi_R(x) = \phi_A(x+q)$,

$$\phi_R(x)|_p = \sum_{m=0}^p \frac{1}{m!} \phi_m x^m, \quad (1.2)$$

which clearly commutes with the observer's momentum. To rephrase any expression in terms of relative fields, we substitute $\phi_A \rightarrow \phi_R$ and $x \rightarrow x+q$. This step explicitly brings out observer-dependence. In particular, the multi-dimensional Virasoro cocycles are non-central because they are functionals of the observer's trajectory $q(t)$, which does not commute with diffeomorphisms.

The results in [7] were formulated directly in terms of the p -jet data, i.e. the Taylor coefficients ϕ_m , the observer's position q , and their canonical momenta. To facilitate calculations and make formulas resemble the corresponding field equations, in the present paper we work with the full

MacLaurin series (1.2) rather than the individual Taylor coefficients. However, the underlying p -jet structure is still present, and manifests itself in some places. We need to replace the d -dimensional delta function $\delta(x - y)$ with the corresponding p -jet delta function $\delta_p(x, y)$. Unlike the ordinary delta function, this object can be multiplied with itself in a meaningful way. That the square of the ordinary delta function is infinite is one reason why the multi-dimensional Virasoro algebra can not be obtained by quantizing the original fields.

The passage to p -jets can be viewed from a slightly different perspective. Introduce a nilpotent number ϵ satisfying $\epsilon^p = 0$, replace all coordinates x^μ with ϵx^μ , and set $\epsilon^m = 1$ for all $m \leq p$ at the end. This procedure automatically picks out the p -jet part, since $\phi_R(\epsilon x) = \phi_R(\epsilon x)|_p$. Moreover, the product of p -jets is handled correctly, because

$$\phi_R(\epsilon x)|_p \psi_R(\epsilon y)|_p = \left(\phi_R(\epsilon x) \psi_R(\epsilon y) \right) \Big|_p. \quad (1.3)$$

The use of nilpotent numbers is not necessary for evaluating bilinear products of operators, because the canonical momentum will pick out the correct p -jet part of products of fields anyway, but the method may be useful in other contexts.

A separate issue with [7] is that normal ordering was carried out using a slightly non-standard formalism. In the present paper we use the standard operator product expansion (OPE) instead, following the notation of [4]. This method is much simpler and should hopefully make the results more accessible.

Before addressing the full diffeomorphism algebra, we start with the simpler case of gauge transformations in Yang-Mills type theories, i.e. the current algebra in d dimensions. It admits a non-trivial extension, which we call the multi-dimensional affine algebra $Aff(d, \mathfrak{g})$. The same algebra is called the *central extension* in [5, 9], but we avoid this name because the extension does not commute with diffeomorphisms.

This paper is organized as follows. In the next section some necessary formalism is established. $Aff(d, \mathfrak{g})$ and its off-shell representations are discussed in section 3. The more complicated case of $Vir(d)$ is dealt with in the section thereafter. In section 5 we note that the representations also admit an intertwining action of an additional Virasoro algebra, which can be identified as the algebra of reparameterizations of the observer's trajectory. We can thus enlarge the symmetry to include reparametrizations "for free", i.e. without enlarging the modules. The main results are summarized in section 6. The final section contains a brief discussion of the relevance of these

algebras to physics. Heavy calculations are relegated to the appendices.

2 p -jet preliminaries

To do calculations with p -jets it is useful to introduce multi-indices. Let $\mathbf{m} = (m_0, m_1, \dots, m_{d-1})$, all $m_\mu \geq 0$, be a multi-index of length $|\mathbf{m}| = \sum_{\mu=0}^{d-1} m_\mu$. The factorial is $\mathbf{m}! = m_0! m_1! \dots m_{d-1}!$, and the power is

$$x^{\mathbf{m}} = (x^0)^{m_0} (x^1)^{m_1} \dots (x^{d-1})^{m_{d-1}}. \quad (2.1)$$

Multi-indices can be added and subtracted,

$$\mathbf{m} + \mathbf{n} = (m_0 + n_0, m_1 + n_1, \dots, m_{d-1} + n_{d-1}). \quad (2.2)$$

The binomial coefficients are defined in the obvious way, i.e. $\binom{\mathbf{m}}{\mathbf{n}} = \mathbf{m}! / \mathbf{n}! ((\mathbf{m} - \mathbf{n})!$. Finally, let μ also denote the unit multi-index in the μ :th direction, i.e. $\mu_\nu = \delta_{\mu,\nu}$, and hence $(\mathbf{m} + \mu)! = m_\mu \mathbf{m}!$.

Let $\phi_A(x)$ is an *absolute field*, where the coordinates $x = (x^\mu)$ are defined relative some fixed but arbitrary global origin. A p -jet is the Taylor expansion of a field $\phi_A(x)$ around the observer's position q^μ , truncated at order p :

$$\phi_A(x)|_p = \sum_{|\mathbf{m}| \leq p} \frac{1}{\mathbf{m}!} \phi_{\mathbf{m}}(x - q)^{\mathbf{m}}. \quad (2.3)$$

The space of p -jets is spanned by the Taylor coefficients $\phi_{\mathbf{m}}$, $|\mathbf{m}| \leq p$, and the expansion point q^μ .

Introduce the corresponding canonical momenta $\pi^{\mathbf{m}}$ and p_μ , satisfying the canonical commutation relations (CCR):

$$[\phi_{\mathbf{m}}, \pi^{\mathbf{n}}] = i\delta_{\mathbf{m}}^{\mathbf{n}}, \quad [\phi_{\mathbf{m}}, \phi_{\mathbf{n}}] = [\pi^{\mathbf{m}}, \pi^{\mathbf{n}}] = 0, \quad (2.4)$$

and

$$[q^\mu, p_\nu] = i\delta_\nu^\mu, \quad [q^\mu, q^\nu] = [p_\mu, p_\nu] = 0. \quad (2.5)$$

The observer's momentum p_μ does not commute with absolute fields, because

$$\begin{aligned} [p_\mu, \phi_A(x)] &= \sum_{|\mathbf{m}| \leq p} \frac{i}{(\mathbf{m} - \mu)!} \phi_{\mathbf{m}}(x - q)^{\mathbf{m} - \mu} \\ &= i\partial_\mu \phi_A(x). \end{aligned} \quad (2.6)$$

To obtain an independent set of canonical variables, we introduce the corresponding *relative field* $\phi_R(x)$, where the coordinates x are measured relative to the observer's position q^μ , rather than relative to some fixed global origin. Define

$$\phi_R(x) \equiv \phi_A(x + q), \quad (2.7)$$

and hence

$$\phi_A(x) = \phi_R(x - q). \quad (2.8)$$

The Taylor expansion of the relative field takes the form of a MacLaurin series:

$$\phi_R(x) = \phi_R(x)|_p = \sum_{|\mathbf{m}| \leq p} \frac{1}{\mathbf{m}!} \phi_{\mathbf{m}} x^{\mathbf{m}}. \quad (2.9)$$

For any field $A(x)$, we use the notation $A(x)|_p$ to denote the corresponding p -jet.

Whereas (2.9) can be viewed as a generating function for the Taylor coefficient $\phi_{\mathbf{m}}$, the natural generating function for the jet momenta is a field in momentum space:

$$\hat{\pi}_R(k) = \sum_{|\mathbf{m}| \leq p} i^{|\mathbf{m}|} \pi^{\mathbf{m}} k^{\mathbf{m}}. \quad (2.10)$$

For infinite jets, the CCR (2.4) becomes

$$[\phi_R(x), \hat{\pi}_R(k)] = i \exp(ik \cdot x), \quad (p = \infty). \quad (2.11)$$

Making an inverse Fourier transform of (2.10) we obtain the following expression for the canonical momentum in position space:

$$\pi_R(x) = \sum_{|\mathbf{m}| \leq p} (-)^{|\mathbf{m}|} \pi^{\mathbf{m}} \partial_{\mathbf{m}} \delta(x) \quad (2.12)$$

The CCR in position space are thus

$$[\phi_R(x), \pi_R(y)] = i \delta_p(x, y), \quad (2.13)$$

where

$$\delta_p(x, y) = \sum_{|\mathbf{m}| \leq p} \frac{(-)^{|\mathbf{m}|}}{\mathbf{m}!} x^{\mathbf{m}} \partial_{\mathbf{m}} \delta(y). \quad (2.14)$$

is the *p-jet delta function*. This is not symmetric,

$$\delta_p(y, x) \neq \delta_p(x, y), \quad (2.15)$$

and it has the following smearing properties, which are proven in Appendix A:

$$\int d^d y f(y) \delta_p(x, y) = f(x)|_p, \quad (2.16)$$

$$\int d^d y f(y) \partial_\mu^x \delta_p(x, y) = \partial_\mu f(x)|_p, \quad (2.17)$$

$$\int d^d y f(y) \partial_\mu^y \delta_p(x, y) = -\partial_\mu f(x)|_p. \quad (2.18)$$

for every smearing function $f(x)$.

A crucial difference between the *p-jet delta function* and the ordinary field delta function is that the product of the former with itself makes sense. In appendix C we prove the following crucial results:

$$\begin{aligned} \delta_p(x, y) \delta_p(y, x) &= \binom{d+p}{d} \delta(x) \delta(y), \\ \partial_\mu^x \delta_p(x, y) \delta_p(y, x) &\approx -\binom{d+p}{d+1} \partial_\mu \delta(x) \delta(y), \\ \partial_\mu^x \delta_p(x, y) \partial_\nu^y \delta_p(y, x) &\approx \binom{d+p+1}{d+2} \partial_\nu \delta(x) \partial_\mu \delta(y) \\ &\quad + \binom{d+p}{d+2} \partial_\mu \delta(x) \partial_\nu \delta(y). \end{aligned} \quad (2.19)$$

The first relation is a strict equality, but the two latter must be understood in a weaker sense. They become equalities if the RHS is smeared with a function $f(x)$ and the LHS is smeared with the function $f_0(x) = f(x) - f(0)$; in the last relation an analogous subtraction is also necessary for the smearing function $g(y)$. Fortunately, it is exactly this modified smearing that is necessary for our purposes.

From the definition (2.12) and an integration by parts it follows that the canonical momentum picks out the *p-jet* part of any field:

$$\int d^d x \pi_R(x) A(x) = \int d^d x \pi_R(x) A(x)|_p. \quad (2.20)$$

Another useful property of *p-jets* is that

$$(A(x)|_p B(x)|_p)|_p = (A(x) B(x))|_p. \quad (2.21)$$

To prove (2.21), we observe that both sides equal the double sum

$$\sum_{\mathbf{m}} \sum_{\mathbf{n}} \frac{1}{\mathbf{m}!\mathbf{n}!} A_{\mathbf{m}} B_{\mathbf{n}} x^{\mathbf{m}+\mathbf{n}}, \quad (2.22)$$

but the summation ranges seem different. In the LHS, the sum runs over \mathbf{m} and \mathbf{n} which satisfy the joint condition $|\mathbf{m}| \leq p$, $|\mathbf{n}| \leq p$, and $|\mathbf{m} + \mathbf{n}| \leq p$, whereas in the RHS the only condition is $|\mathbf{m} + \mathbf{n}| \leq p$. However, the two first conditions in the LHS are in fact redundant, because if $|\mathbf{m} + \mathbf{n}| \leq p$, then $|\mathbf{m}|$ and $|\mathbf{n}|$ are both automatically $\leq p$. Hence the two sides of (2.21) are indeed equal.

3 Multi-dimensional affine algebra

3.1 Classical representations

Let \mathfrak{g} be a Lie algebra with basis J^a , totally anti-symmetric structure constants f^{abc} , and Killing metric δ^{ab} . Due to the existence of the Killing metric, there is no need to distinguish between upper and lower \mathfrak{g} indices. The \mathfrak{g} brackets are

$$[J^a, J^b] = i f^{abc} J^c. \quad (3.1)$$

Let the matrices M^a form a basis for a finite-dimensional \mathfrak{g} representation M , and let y_M be the value of the second Casimir operator in this representation, defined by

$$\text{tr } M^a M^b = y_M \delta^{ab}. \quad (3.2)$$

The algebra of maps from the d -dimensional base manifold into \mathfrak{g} , $\mathbf{map}(d, \mathfrak{g})$, is defined by the brackets

$$[\mathcal{J}_X, \mathcal{J}_Y] = \mathcal{J}_{[X, Y]}, \quad (3.3)$$

where $X = X^a(x) J^a$ is a \mathfrak{g} -valued function and $[X, Y] = i f^{abc} X^a Y^b J^c$.

For each finite-dimensional \mathfrak{g} representation M , the corresponding $\mathbf{map}(d, \mathfrak{g})$ representation acts on M -valued fields:

$$[\mathcal{J}_X, \phi(x)] = -X^a(x) M^a \phi(x) \equiv -X(x) \phi(x). \quad (3.4)$$

Let $\pi(x)$ be the canonical conjugate of $\phi(x)$, defined by the Heisenberg algebra

$$\begin{aligned} [\phi(x), \pi(y)] &= i \delta(x - y), \\ [\phi(x), \phi(y)] &= [\pi(x), \pi(y)] = 0. \end{aligned} \quad (3.5)$$

$\pi(x)$ transforms in the dual representation:

$$[\mathcal{J}_X, \pi(x)] = X^a(x)\pi(x)M^a = \pi(x)X(x). \quad (3.6)$$

Hence the current algebra is generated by the operators

$$\mathcal{J}_X = -i \int d^d x \pi(x)X(x)\phi(x). \quad (3.7)$$

3.2 Quantization in one dimension

Let us first review how to build representations of the ordinary affine algebra $Aff(1)$, using the OPE formalism [4]. Let $\phi(z)$ be an M -valued field and let $\pi(z)$ be its canonical momentum. The OPEs of the fundamental fields read

$$\begin{aligned} \phi(z)\pi(w) &\sim \frac{1}{z-w}, \\ \phi(z)\phi(w) &\sim \pi(z)\pi(w) \sim 0. \end{aligned} \quad (3.8)$$

We denote the z derivative by a dot, $\dot{\phi}(z) \equiv d\phi(z)/dz$, to distinguish it from derivatives w.r.t. spacetime coordinates below. Note that we have not identified momenta and velocities, $\pi(z) \neq \dot{\phi}(z)$. In string theory fields obeying the definition (3.8) are often called ghosts, cf [4], eqn. (6.238).

Equation (3.8) holds both for bosonic and fermionic fields, but the OPE of the fields in opposite order depends on Grassman parity:

$$\begin{aligned} \pi(z)\phi(w) &\sim -\frac{1}{z-w}, & \text{bosonic} \\ \pi(z)\phi(w) &\sim \frac{1}{z-w}, & \text{fermionic.} \end{aligned} \quad (3.9)$$

To treat both bosons and fermions at the same time, we summarize the OPE of the fundamental fields as

$$\phi(z)\pi(w) \sim \mp \pi(z)\phi(w) \sim \frac{1}{z-w}. \quad (3.10)$$

Here and henceforth the upper sign refers to bosons and the lower sign to fermions.

The generators of $Aff(1, \mathfrak{g})$ are

$$J^a(z) =: \pi(z)M^a\phi(z) :, \quad (3.11)$$

where double dots denotes normal ordering. The field transforms in M and the canonical momentum transforms in the dual representation:

$$\begin{aligned} J^a(z)\phi(w) &\sim -\frac{M^a\phi(w)}{z-w}, \\ J^a(z)\pi(w) &\sim \frac{\pi(w)M^a}{z-w}, \end{aligned} \tag{3.12}$$

The affine algebra

$$J^a(z)J^b(w) \sim if^{abc}\frac{J^c(w)}{z-w} + \frac{k\delta^{ab}}{(z-w)^2}, \tag{3.13}$$

where the central charge $k = \mp y_M$ is given by the value of the second Casimir in (3.2). In terms of the smeared generators

$$\mathcal{J}_X = \frac{1}{2\pi i} \oint dz X^a(z) J^a(z), \tag{3.14}$$

the affine algebra takes the form

$$\begin{aligned} [\mathcal{J}_X, \mathcal{J}_Y] &= \oint_0 dw \oint_w dz J^a(z) J^b(w) \\ &= \mathcal{J}_{[X,Y]} + \frac{k}{2\pi i} \oint dw \dot{X}^a(w) Y(w). \end{aligned} \tag{3.15}$$

3.3 Naïve field quantization

Let us now attempt to repeat the construction in the previous subsection for $Aff(d, \mathfrak{g})$. In several dimensions there is no preferred time direction that defines the normal order. One could define a lexicographical order as is done for finite-dimensional Lie algebras, but this is problematic for several reasons, e.g. because there is no guarantee that different orderings will lead to equivalent representations; the Stone-von Neumann theorem does not hold in infinite dimensions.

Instead, we explicitly introduce an extra coordinate z to define normal ordering. One way to think about this is that we have fixed a foliation, t is the time coordinate, and the complex coordinate $z = \exp(it)$; x denotes the spatial coordinates. $\mathfrak{map}(d, \mathfrak{g})$ is then the algebra of spatial gauge transformations. Another interpretation is that t is a time-like parameter along the observer's trajectory, which should be identified with the time coordinate x^0 at a later stage.

Either way, the fundamental fields now depend on an extra complex variable, and the OPEs become

$$\begin{aligned}\phi(x, z)\pi(y, w) &\sim \mp \pi(x, z)\phi(y, w) \sim \frac{1}{z-w} \delta(x-y), \\ \phi(x, z)\phi(y, w) &\sim \pi(x, z)\pi(y, w) \sim 0.\end{aligned}\tag{3.16}$$

The smeared generators,

$$J_X(z) = \int d^d x : \pi(x, z) X(x, z) \phi(x, z) :, \tag{3.17}$$

act as follows on the fundamental fields

$$\begin{aligned}J_X(z)\phi(x, w) &\sim \frac{-1}{z-w} X(x, w)\phi(x, w), \\ J_X(z)\pi(x, w) &\sim \frac{1}{z-w} \pi(x, w) X(x, w).\end{aligned}\tag{3.18}$$

We now encounter a serious problem with the extension: the central charge becomes infinite. Formally,

$$\begin{aligned}J_X(z)J_Y(w) &\sim \frac{1}{z-w} J_{[X,Y]}(w) \\ &\mp \frac{1}{(z-w)^2} \iint d^d x d^d y \operatorname{tr}(X(x, z) Y(y, w)) \delta(x-y) \delta(y-x).\end{aligned}\tag{3.19}$$

The double Wick contraction is proportional to $\delta(0)$, since

$$\delta(x-y)\delta(y-x) = \delta(0)\delta(x-y), \tag{3.20}$$

and hence the central charge is infinite,

$$k = \mp y_M \delta(0). \tag{3.21}$$

Clearly, an infinite central charge does not make sense. The generalization of the affine algebra to higher dimensions must be done differently, a task to which we now turn.

3.4 p -jet quantization

To overcome the problem with the infinite central charge, we replace fields with the corresponding p -jets. To maintain as much as possible of the field formalism, the Taylor coefficients are not displayed explicitly, but instead

everything is expressed in terms of the generating functions (2.9) and (2.12). The OPE of the fundamental fields remains essentially unchanged; the only difference compared to (3.16) is that the delta function has been replaced by its truncation $\delta_p(x, y)$,

$$\begin{aligned}\phi(x, z)\pi(y, w) &\sim \frac{1}{z-w} \delta_p(x, y), \\ \pi(x, z)\phi(y, w) &\sim \frac{\mp 1}{z-w} \delta_p(y, x)\end{aligned}\tag{3.22}$$

Recall that the p -jet delta function is not symmetric. Because we deal with relative fields,

$$X(x, z) = X(x + q(z)),\tag{3.23}$$

so the smeared generators (3.17) are replaced by

$$J_X(z) = \int d^d x : \pi(x, z) X(x + q(z)) \phi(x, z) : .\tag{3.24}$$

To see that this expression is indeed equal to the gauge algebra generators in [7], we use the definitions (2.9) and (2.12) and suppress the z dependence. First integrate repeatedly by parts:

$$\begin{aligned}J_X &= \int : \left(\sum_{|\mathbf{m}| \leq p} (-)^{|\mathbf{m}|} \pi^{\mathbf{m}} \partial_{\mathbf{m}} \delta(x) \right) \left(\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \partial_{\mathbf{r}} X(q) x^{\mathbf{r}} \right) \left(\sum_{|\mathbf{n}| \leq p} \frac{1}{\mathbf{n}!} \phi_{\mathbf{n}} x^{\mathbf{n}} \right) : \\ &= \sum \sum \sum \frac{1}{\mathbf{n}! \mathbf{r}!} : \pi^{\mathbf{m}} \partial_{\mathbf{r}} X(q) \phi_{\mathbf{n}} : \int d^d x \partial_{\mathbf{m}} (x^{\mathbf{n}+\mathbf{r}}) \delta(x).\end{aligned}\tag{3.25}$$

The last integral equals $\mathbf{m}! \delta_{\mathbf{m}}^{\mathbf{n}+\mathbf{r}}$, and hence

$$J_X = \sum_{|\mathbf{m}| \leq p} \sum_{|\mathbf{n}| \leq p} \pi^{\mathbf{m}} J_{\mathbf{m}}^{\mathbf{n}}(X(q)) \phi_{\mathbf{m}},\tag{3.26}$$

where

$$J_{\mathbf{m}}^{\mathbf{n}}(X) = \binom{\mathbf{m}}{\mathbf{n}} \partial_{\mathbf{m}-\mathbf{n}} X.\tag{3.27}$$

Since the binomial coefficients vanish whenever $\mathbf{n} < \mathbf{m}$, the sum over \mathbf{n} is in fact restricted to this range. Equations (3.26) and (3.27) are thus equal to equations (6.6) and (6.4) of [7], respectively.

The OPE $J_X(z)J_Y(w)$ becomes

$$\begin{aligned}
& \frac{1}{z-w} \iint d^d x d^d y \delta_p(x, y) : \pi(x, z) X(x, z) Y(y, w) \phi(y, w) : \\
& - \frac{1}{z-w} \iint d^d x d^d y \delta_p(y, x) : \pi(y, w) Y(y, w) X(x, z) \phi(x, z) : \\
& \mp \frac{1}{(z-w)^2} \iint d^d x d^d y \delta_p(x, y) \delta_p(y, x) \text{tr}(X(x, z) Y(y, w)).
\end{aligned} \tag{3.28}$$

We must be careful when evaluating this expression, because $\delta_p(x, y) \neq \delta_p(y, x)$. Suppressing normal ordering and the z and w dependence, the inner integral in the first term can be rewritten as

$$\begin{aligned}
& \int d^d y \delta_p(x, y) \pi(x) X(x) Y(y) \phi(y) \\
& = \pi(x) X(x) (Y(x) \phi(x))|_p \\
& = \pi(x) (X(x) (Y(x) \phi(x))|_p)|_p \\
& = \pi(x) (X(x) Y(x) \phi(x))|_p,
\end{aligned} \tag{3.29}$$

where we used the properties (2.20) and (2.21). Finally using (2.20) once again, the first term in (3.28) becomes

$$\frac{1}{z-w} \int d^d x : \pi(x, z) X(x, z) Y(x, z) \phi(x, z) : \tag{3.30}$$

The double Wick contraction is calculated using (C.1), and equals

$$\begin{aligned}
& \frac{\mp 1}{(z-w)^2} \int d^d x d^d y A_{d,p} \delta(x) \delta(y) \text{tr}(X(x, z) Y(y, w)) \\
& = \frac{\mp 1}{(z-w)^2} \binom{d+p}{d} \text{tr}(X(0, z) Y(0, w)),
\end{aligned} \tag{3.31}$$

where we used (B.1) in the last step.

Hence the OPE is

$$\begin{aligned}
J_X(z) J_Y(w) & \sim \frac{1}{z-w} J_{[X,Y]}(w) \\
& \mp \frac{1}{(z-w)^2} \binom{d+p}{d} \text{tr}(X(0, z) Y(0, w)).
\end{aligned} \tag{3.32}$$

Equivalently, the operators

$$\mathcal{J}_X = \frac{1}{2\pi i} \oint dz J_X(z) \tag{3.33}$$

satisfy the Lie algebra brackets

$$\begin{aligned} [\mathcal{J}_X, \mathcal{J}_Y] &= \frac{-1}{4\pi} \oint_0 dw \oint_w dz J_X(z) J_Y(w) \\ &= \mathcal{J}_{[X,Y]} + \frac{k}{2\pi i} \oint dw \dot{X}^a(0, w) Y^a(0, w). \end{aligned} \quad (3.34)$$

where

$$k = \mp \binom{d+p}{d} y_M, \quad (3.35)$$

and the dot denotes the partial derivative w.r.t. the complex coordinate, $\dot{f}(x, z) \equiv \partial f(x, z)/\partial z$. In particular, with $X(x, z) = X(x + q(z))$,

$$[\mathcal{J}_X, \mathcal{J}_Y] = \mathcal{J}_{[X,Y]} + \frac{k}{2\pi i} \oint dw \dot{q}^\mu(w) \partial_\mu X^a(q(w)) Y^a(q(w)), \quad (3.36)$$

which is the form of $Aff(d, \mathfrak{g})$ described in [7].

Note that this cocycle is proportional to the second Casimir operator. $Aff(d, \mathfrak{g})$ is thus unrelated to the gauge anomalies appearing in the standard model of particle physics, which are proportional to the third Casimir.

In one dimension, the cocycle can be rewritten as $\int dq X'_a(q) Y^a(q)$, which shows that (3.36) reduces to the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ when $d = 1$.

4 Multi-dimensional Virasoro algebra

4.1 Classical representations

Let $\xi = \xi^\mu(x) \partial_\mu$ be a vector field, with commutator

$$[\xi, \eta] \equiv \xi^\mu \partial_\mu \eta^\nu \partial_\nu - \eta^\nu \partial_\nu \xi^\mu \partial_\mu. \quad (4.1)$$

The algebra of vector fields in d dimensions, denoted by $\mathbf{vect}(d)$, is generated by the Lie derivatives \mathcal{L}_ξ . The bracket reads

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi, \eta]}. \quad (4.2)$$

We will colloquially refer to $\mathbf{vect}(d)$ as the d -dimensional diffeomorphism algebra, although it is not strictly the Lie algebra of the diffeomorphism group in d dimensions.

The most natural type of $\mathbf{vect}(d)$ representations act on modules of tensor densities. In one dimension, these are the primary fields of CFT. Consider

the Heisenberg algebra generated by a tensor-valued spacetime field $\phi(x)$ and its canonical momentum $\pi(x)$:

$$[\phi(x), \pi(y)] = i\delta(x - y), \quad [\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0. \quad (4.3)$$

$\mathbf{vect}(d)$ can be embedded into this Heisenberg algebra as follows:

$$\mathcal{L}_\xi = -i \int d^d x \left(\xi^\mu(x) \pi(x) \partial_\mu \phi(x) + \partial_\nu \xi^\mu(x) \pi(x) T_\mu^\nu \phi(x) \right), \quad (4.4)$$

where the matrices T_ν^μ satisfy $\mathfrak{gl}(d)$:

$$[T_\rho^\mu, T_\sigma^\nu] = \delta_\rho^\nu T_\sigma^\mu - \delta_\sigma^\mu T_\rho^\nu. \quad (4.5)$$

For every $\mathfrak{gl}(d)$ representation ϱ , the embedding (4.4) yields a $\mathbf{vect}(d)$ representation acting on tensor densities of type ϱ :

$$[\mathcal{L}_\xi, \phi(x)] = -\xi^\mu(x) \partial_\mu \phi(x) - \partial_\nu \xi^\mu(x) T_\mu^\nu \phi(x). \quad (4.6)$$

The conjugate momentum transforms as a density in the dual representation:

$$[\mathcal{L}_\xi, \pi(x)] = -\xi^\mu(x) \partial_\mu \pi(x) - \partial_\mu \xi^\mu(x) \pi(x) + \partial_\nu \xi^\mu(x) \pi(x) T_\mu^\nu. \quad (4.7)$$

4.2 Relative fields

The fields in the classical embedding (4.4) are absolute fields, To prepare for quantization, we follow the same recipe as for gauge algebras; replace absolute fields with relative fields and shift $x^\mu \rightarrow x^\mu + q^\mu$ to make the dependence on the observer's position manifest. This leads to

$$\mathcal{L}_\xi^\phi = -i \int d^d x \left(\xi^\mu(x + q) \pi(x) \partial_\mu \phi(x) + \partial_\nu \xi^\mu(x + q) \pi(x) T_\mu^\nu \phi(x) \right), \quad (4.8)$$

However, diffeomorphisms do not commute with the observer's position. Hence we also need a second class of $\mathbf{vect}(d)$ representations, acting nonlinearly on q^μ . The embedding of $\mathbf{vect}(d)$ is given by

$$\mathcal{L}_\xi^q = i\xi^\mu(q) p_\mu, \quad (4.9)$$

where the $2d$ generators q^μ and p_ν satisfy the CCR (2.5). The action on the observer's position is non-linear:

$$\begin{aligned} [\mathcal{L}_\xi^q, q^\mu] &= \xi^\mu(q), \\ [\mathcal{L}_\xi^q, p_\nu] &= -\partial_\nu \xi^\mu(q) p_\mu. \end{aligned} \quad (4.10)$$

These relations are to be understood as defining nonlinear realizations in the space of polynomial functions of q^μ . If $\Phi(q)$ is such a function, the operators \mathcal{L}_ξ^q act as scalar fields:

$$[\mathcal{L}_\xi^q, \Phi(q)] = \xi^\mu(q) \partial_\mu \Phi(q). \quad (4.11)$$

The operators in (4.8) and (4.9) both satisfy $\mathbf{vect}(d)$ individually, but their sum $\mathcal{L}_\xi^\phi + \mathcal{L}_\xi^q$ does not, because the observer's momentum does not commute with $\xi^\mu(x+q)$. To remedy this defect, we introduce the improved momentum

$$P_\mu = p_\mu + \int d^d x \pi(x) \partial_\mu \phi(x), \quad (4.12)$$

which satisfies

$$\begin{aligned} [P_\mu, q^\nu] &= -i\delta_\mu^\nu, \\ [P_\mu, \phi(x)] &= -i\partial_\mu \phi(x), \\ [P_\mu, \pi(x)] &= -i\partial_\mu \pi(x), \\ [P_\mu, P_\nu] &= 0. \end{aligned} \quad (4.13)$$

The improved generators

$$\mathcal{L}_\xi'^q = \xi^\mu(q) P_\mu \quad (4.14)$$

satisfy $[\mathcal{L}_\xi'^q, \mathcal{L}_\eta'^q] = \mathcal{L}_{[\xi, \eta]}'^q$ and $[\mathcal{L}_\xi'^q, \mathcal{L}_\eta^\phi] = 0$. The sum

$$\mathcal{L}_\xi = \mathcal{L}_\xi'^q + \mathcal{L}_\xi^\phi \quad (4.15)$$

hence furnishes a realization of $\mathbf{vect}(d)$ which acts correctly on both the observer's position and on the relative fields.

4.3 Quantization

The OPEs between the fundamental fields are defined to be

$$\begin{aligned} q^\mu(z) p_\nu(w) &\sim \frac{1}{z-w} \delta_\nu^\mu, \\ \phi(x, z) \pi(y, w) &\sim \frac{1}{z-w} \delta_p(x, y), \end{aligned} \quad (4.16)$$

whereas all other contractions vanish.

It is useful to arrange the diffeomorphism generators somewhat differently than in (4.15). Set

$$\begin{aligned}
L_\xi^0(z) &= - : \xi^\mu(q(z)) p_\mu(z) : \\
L_\xi^1(z) &= \int d^d x (\xi^\mu(x + q(z)) - \xi^\mu(q(z))) : \pi(x, z) \partial_\mu \phi(x, z) :, \\
L_\xi^2(z) &= \int d^d x \partial_\nu \xi^\mu(x + q(z)) : \pi(x, z) T_\mu^\nu \phi(x, z) : .
\end{aligned} \tag{4.17}$$

The total $Vir(d)$ generators are

$$L_\xi(z) = L_\xi^0(z) + L_\xi^1(z) + L_\xi^2(z). \tag{4.18}$$

The OPEs with the fundamental fields are

$$\begin{aligned}
L_\xi(z) q^\mu(w) &\sim \frac{1}{z-w} \xi^\mu(q(w)), \\
L_\xi(z) p_\nu(w) &\sim \frac{1}{z-w} \left(- \partial_\nu \xi^\mu(q(w)) p_\mu(w) \right. \\
&\quad + \int d^d x (\partial_\nu \xi^\mu(x + q(w)) - \partial_\nu \xi^\mu(q(w))) : \pi(x, w) \partial_\mu \phi(x, w) : \\
&\quad \left. + \int d^d x \partial_\nu \partial_\rho \xi^\mu(x + q(w)) : \pi(x, w) T_\mu^\rho \phi(x, w) : \right), \\
L_\xi(z) \phi(x, w) &\sim \frac{1}{z-w} \left(- (\xi^\mu(x + q(w)) - \xi^\mu(q(w))) \partial_\mu \phi(x, w) \right. \\
&\quad \left. - \partial_\nu \xi^\mu(x + q(w)) T_\mu^\nu \phi(x, w), \right) \\
L_\xi(z) \pi(x, w) &\sim \frac{1}{z-w} \left(- (\xi^\mu(x + q(w)) - \xi^\mu(q(w))) \partial_\mu \pi(x, w) \right. \\
&\quad \left. + \partial_\nu \xi^\mu(x + q(w)) \pi(x, w) T_\mu^\nu, \right).
\end{aligned} \tag{4.19}$$

To calculate the OPE $L_\xi(z) L_\eta(w)$ is quite tedious, and is deferred to Appendix D. The result is

$$L_\xi(z) L_\eta(w) \sim \frac{L_{[\xi, \eta]}(w)}{z-w} + \frac{Z_{\xi, \eta}(z, w)}{(z-w)^2}, \tag{4.20}$$

where

$$\begin{aligned}
Z_{\xi, \eta}(z, w) &= - c_1 \partial_\nu \xi^\mu(q(z)) \partial_\mu \eta^\nu(q(w)) \\
&\quad - c_2 \partial_\mu \xi^\mu(q(z)) \partial_\nu \eta^\nu(q(w)).
\end{aligned} \tag{4.21}$$

Assume that the fields transform in a representation ϱ of $\mathfrak{gl}(d)$, and let I be the unit matrix in this representation.

$$\begin{aligned}\mathrm{tr} I &= \dim \varrho \equiv \Delta_\varrho, \\ \mathrm{tr} T_\nu^\mu &= k_0(\varrho) \delta_\nu^\mu, \\ \mathrm{tr} T_\rho^\mu T_\sigma^\nu &= k_1(\varrho) \delta_\rho^\mu \delta_\sigma^\nu + k_2(\varrho) \delta_\sigma^\mu \delta_\rho^\nu.\end{aligned}\tag{4.22}$$

The *abelian charges* c_1 and c_2 are given by

$$\begin{aligned}c_1 &= 1 \pm \left\{ \binom{d+p+1}{d+2} \Delta_\varrho + \binom{d+p}{d} k_1(\varrho) \right\}, \\ c_2 &= \pm \left\{ \binom{d+p}{d+2} \Delta_\varrho + 2 \binom{d+p}{d+1} k_0(\varrho) + \binom{d+p}{d} k_2(\varrho) \right\},\end{aligned}\tag{4.23}$$

where the upper sign refers to bosonic fields and the lower to fermionic fields.

The $\mathfrak{gl}(d)$ parameters (4.22) can be rewritten using more conventional notation if we note that $\mathfrak{gl}(d) = \mathfrak{sl}(d) \oplus \mathfrak{gl}(1)$. A $\mathfrak{gl}(d)$ matrix is of the form

$$T_\nu^\mu = S_\nu^\mu + \kappa \delta_\nu^\mu I,\tag{4.24}$$

where S_ν^μ is an $\mathfrak{sl}(d)$ matrix and κ is the weight of the field $\phi(x, z)$ as a density. By definition,

$$S_\mu^\mu = 0.\tag{4.25}$$

$\mathfrak{gl}(d)$ representations are labelled by a pair (ϱ, κ) , where ϱ now is an $\mathfrak{sl}(d)$ representation. The $\mathfrak{sl}(d)$ traces are

$$\begin{aligned}\mathrm{tr} S_\nu^\mu &= 0, \\ \mathrm{tr} S_\rho^\mu S_\sigma^\nu &= y_\varrho (\delta_\sigma^\mu \delta_\rho^\nu - \frac{1}{d} \delta_\rho^\mu \delta_\sigma^\nu),\end{aligned}\tag{4.26}$$

where y_ϱ is the value of the quadratic Casimir in ϱ . The last condition guarantees that $\mathrm{tr} S_\mu^\mu S_\sigma^\nu = 0$. The $\mathfrak{gl}(d)$ traces can now be written as

$$\begin{aligned}\mathrm{tr} I &= \Delta_\varrho, \\ \mathrm{tr} T_\nu^\mu &= \kappa \Delta_\varrho \delta_\nu^\mu, \\ \mathrm{tr} T_\rho^\mu T_\sigma^\nu &= y_\varrho \delta_\sigma^\mu \delta_\rho^\nu + (\kappa^2 \Delta_\varrho - \frac{y_\varrho}{d}) \delta_\rho^\mu \delta_\sigma^\nu.\end{aligned}\tag{4.27}$$

Hence

$$\begin{aligned}k_0(\varrho) &= \kappa \Delta_\varrho, \\ k_1(\varrho) &= y_\varrho, \\ k_2(\varrho) &= \kappa^2 \Delta_\varrho - \frac{y_\varrho}{d}.\end{aligned}\tag{4.28}$$

Finally we express the OPE (4.20) – (4.21) as a Lie algebra. Define

$$\mathcal{L}_\xi = \frac{1}{2\pi i} \oint dz L_\xi(z). \quad (4.29)$$

These operators satisfy the *multi-dimensional Virasoro algebra* $Vir(d)$, i.e.

$$\begin{aligned} [\mathcal{L}_\xi, \mathcal{L}_\eta] &= \mathcal{L}_{[\xi, \eta]} + \mathcal{Z}_{\xi, \eta}, \\ [\mathcal{L}_\xi, q^\mu(t)] &= \xi^\mu(q(t)), \\ [q^\mu(t), q^\nu(t')] &= 0. \end{aligned} \quad (4.30)$$

where the extension is

$$\begin{aligned} \mathcal{Z}_{\xi, \eta} &= \frac{-1}{2\pi i} \oint dz \left(c_1 \partial_\nu \dot{\xi}^\mu(q(z)) \partial_\mu \eta^\nu(q(z)) + c_2 \partial_\mu \dot{\xi}^\mu(q(z)) \partial_\nu \eta^\nu(q(z)) \right) \\ &= \frac{-1}{2\pi i} \oint dz \dot{q}^\rho(z) \left(c_1 \partial_\rho \partial_\nu \xi^\mu(q(z)) \partial_\mu \eta^\nu(q(z)) \right. \\ &\quad \left. + c_2 \partial_\rho \partial_\mu \xi^\mu(q(z)) \partial_\nu \eta^\nu(q(z)) \right). \end{aligned} \quad (4.31)$$

The cocycle proportional to c_1 was discovered by Rao and Moody [10], and the one proportional to c_2 by myself [6].

4.4 Intertwining action on $Aff(d, \mathfrak{g})$

In this subsection we complete the full semi-direct product $Vir(d) \ltimes Aff(d, \mathfrak{g})$, or more precisely the extension of $\mathbf{vect}(d) \ltimes \mathbf{map}(d, \mathfrak{g})$. There is a distinction because the mixed bracket also acquires an extension, although it vanishes in the case that \mathfrak{g} is semisimple.

The OPE between the operators (4.18) and (3.24) reads

$$L_\xi(z) J_X(w) \sim \frac{J_{\xi X}(w)}{z - w} + \frac{W_{\xi, X}(z, w)}{(z - w)^2}, \quad (4.32)$$

where $X(z)$ transforms as a density of weight one:

$$\xi X = \xi^\mu \partial_\mu X + \partial_\mu \xi^\mu X. \quad (4.33)$$

The extension is non-zero only if the trace does not vanish in the \mathfrak{g} representation M . Assume that the \mathfrak{g} matrices satisfy

$$\mathrm{tr} M^a = z_M \delta^a, \quad (4.34)$$

where δ^a is a privileged matrix in M . The archtypical case is $\mathfrak{g} = \mathfrak{gl}(1)$, where the privileged matrix is unity. If the parameter z_M is nonzero, the extension in (4.32) becomes

$$W_{\xi,X}(z,w) = c_7 \partial_\mu \xi^\mu(q(z)) \delta^a X^a(q(w)). \quad (4.35)$$

The value of the abelian charge c_7 equals

$$c_7 = \mp z_M \left\{ \binom{d+p}{d+1} \Delta_\varrho + \binom{d+p}{d} k_0(\varrho) \right\}, \quad (4.36)$$

where Δ_ϱ and $k_0(\varrho)$ were defined in (4.22).

The OPE (4.32) corresponds to the following bracket between \mathcal{L}_ξ and \mathcal{J}_X (4.29) and (3.33):

$$\begin{aligned} [\mathcal{L}_\xi, \mathcal{J}_X] &= \mathcal{J}_{\xi X} + \frac{c_7}{2\pi i} \oint dz \partial_\mu \dot{\xi}^\mu(q(z)) \delta^a X^a(q(z)) \\ &= \mathcal{J}_{\xi X} + \frac{c_7}{2\pi i} \oint dz \dot{q}^\rho(z) \partial_\rho \partial_\mu \xi^\mu(q(z)) \delta^a X^a(q(z)). \end{aligned} \quad (4.37)$$

Finally we must correct the previously computed abelian charges for the fact that M commutes with $\mathbf{vect}(d)$ and ϱ commutes with $\mathbf{map}(d, \mathfrak{g})$. The fields have hence additional indices on which the various algebras act trivially, each of which contributes an equal amount to the abelian charges. The $Vir(d)$ charges c_1 and c_2 in (4.23) are multiplied with Δ_M , and $Aff(d, \mathfrak{g})$ charge $k = c_5$ in (3.35) is multiplied with Δ_ϱ . This correction is accounted for in the summary in section 6.

The proof of the formulas in this section is given in Appendix E.

5 Reparametrization Virasoro algebra

The fields $\phi(x, z)$ depend not only on the spacetime coordinate x but also on the holomorphic variable z , which was identified as a time-like parameter along the observer's spacetime trajectory in [7]. We can therefore extend an $Vir(d) \ltimes Aff(d, \mathfrak{g})$ representation to a representation of reparametrizations “for free”, i.e. without introducing new field components. After quantization, reparametrizations generate a Virasoro algebra which intertwines with the multi-dimensional Virasoro and affine algebras.

All proofs in this section are carried out in Appendix F.

5.1 The energy-momentum tensor in CFT

The energy-momentum of a conformal field $\phi(z)$ and its conjugate momentum $\pi(z)$ is

$$T(z) = - : \pi(z) \dot{\phi}(z) : + \lambda \frac{d}{dz} (: \pi(z) \phi(z) :), \quad (5.1)$$

where λ is the conformal weight of $\phi(z)$, not to be confused with the weight κ as a tensor density (4.24). The conjugate momentum $\pi(z)$ has conformal weight $1 - \lambda$.

The OPEs between $T(z)$ and the fundamental fields are

$$\begin{aligned} T(z)\phi(w) &\sim \frac{\dot{\phi}(w)}{z-w} + \frac{\lambda}{(z-w)^2} \phi(w), \\ T(z)\pi(w) &\sim \frac{\dot{\pi}(w)}{z-w} + \frac{1-\lambda}{(z-w)^2} \pi(w). \end{aligned} \quad (5.2)$$

In particular, if $\phi(z)$ has conformal weight $\lambda = 1$,

$$\begin{aligned} T(z) &= : \dot{\pi}(z) \phi(z) :, \\ T(z)\phi(w) &\sim \frac{d}{dw} \left(\frac{\phi(w)}{z-w} \right). \end{aligned} \quad (5.3)$$

The OPE of the energy-momentum tensor with itself reads

$$T(z)T(w) \sim \frac{\dot{T}(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4}, \quad (5.4)$$

where c is the central charge.

$$c = \pm 2(6\lambda^2 - 6\lambda + 1), \quad (5.5)$$

where as usual the upper (lower) sign applies to bosonic (fermionic) fields.

5.2 Reparametrization algebra

Reparametrizations act on two types of fields: the observer's trajectory $q^\mu(z)$ and the p -jet fields $\phi(x, z)$, and also on their canonical momenta. The definition (5.1) must therefore be replaced by

$$\begin{aligned} T(z) &= - : \dot{q}^\mu(z) p_\mu(z) : \\ &+ \int d^d x \left(- : \pi(x, z) \dot{\phi}(x, z) : + \lambda \frac{d}{dz} (: \pi(x, z) \phi(x, z) :) \right), \end{aligned} \quad (5.6)$$

The OPEs between the reparametrization generators and the fundamental fields read:

$$\begin{aligned}
T(z)q^\mu(w) &\sim \frac{\dot{q}^\mu(w)}{z-w}, \\
T(z)p_\nu(w) &\sim \frac{d}{dw}\left(\frac{p_\nu(w)}{z-w}\right), \\
T(z)\phi(x, w) &\sim \frac{\dot{\phi}(x, w)}{z-w} + \frac{\lambda}{(z-w)^2}\phi(x, w), \\
T(z)\pi(x, w) &\sim \frac{\dot{\pi}(x, w)}{z-w} + \frac{1-\lambda}{(z-w)^2}\pi(x, w).
\end{aligned} \tag{5.7}$$

The OPE $T(z)T(w)$ is given by (5.4), and the central charge equals

$$c \equiv c_4 = 2d \pm 2(6\lambda^2 - 6\lambda + 1) \binom{d+p}{d} \Delta_\varrho \Delta_M. \tag{5.8}$$

We recognize the contributions from the $2d$ fields $q^\mu(z)$ and $p_\nu(z)$, and from the $\binom{d+p}{d} \Delta_\varrho \Delta_M$ field pairs $\phi(x, z)$ and $\pi(x, z)$. The smeared operators

$$\mathcal{T}_f = -\frac{1}{2\pi i} \oint dz f(z) T(z) \tag{5.9}$$

generate the Virasoro algebra $Vir(1)$:

$$[\mathcal{T}_f, \mathcal{T}_g] = \mathcal{T}_{[f, g]} - \frac{c_4}{24\pi i} \oint dz \ddot{f}(z) \dot{g}(z), \tag{5.10}$$

where $[f, g] = f\dot{g} - g\dot{f}$.

5.3 Intertwining action on $Vir(d)$ and $Aff(d, \mathfrak{g})$

Conformal weights are additive. If the fields $\phi_1(x, z)$ and $\phi_2(x, z)$ have weights λ_1 and λ_2 , the product $\phi_1(x, z)\phi_2(x, z)$ has weight $\lambda_1 + \lambda_2$. In particular, since the observer's trajectory $q^\mu(z)$ has zero weight, so has any function $X(q(z))$. Hence the diffeomorphism and gauge generators (4.18) and (3.24) have weight $(1-\lambda)+0+\lambda = 1$. The OPEs between the reparametrization algebra $Vir(1)$ and $Vir(d)$ and $Aff(d, \mathfrak{g})$ are

$$T(z)L_\xi(w) \sim \frac{d}{dw}\left(\frac{L_\xi(w)}{z-w}\right) + \frac{c_3}{(z-w)^3} \partial_\mu \xi^\mu(q(w)), \tag{5.11}$$

$$T(z)J_X(w) \sim \frac{d}{dw}\left(\frac{J_X(w)}{z-w}\right) + \frac{c_6}{(z-w)^3} \delta^a X^a(q(w)), \tag{5.12}$$

where two new abelian charges were introduced:

$$c_3 = 1 \pm (2\lambda - 1)\Delta_M \left\{ \binom{d+p}{d+1} \Delta_\varrho + \binom{d+p}{d} k_0(\varrho) \right\}, \quad (5.13)$$

$$c_6 = \pm(2\lambda - 1)z_M \binom{d+p}{d} \Delta_\varrho. \quad (5.14)$$

Note the contribution from the observer's position to c_3 . The corresponding Lie brackets between smearing operators are

$$\begin{aligned} [\mathcal{T}_f, \mathcal{L}_\xi] &= -\frac{c_3}{4\pi i} \oint dz \ddot{f}(z) \partial_\mu \xi^\mu(q(z)), \\ [\mathcal{T}_f, \mathcal{J}_X] &= -\frac{c_6}{4\pi i} \oint dz \ddot{f}(z) \delta^a X^a(q(z)). \end{aligned} \quad (5.15)$$

6 Summary of the main formulas

In this section we collect the main equations for easy reference.

Operator product expansions:

$$\begin{aligned} L_\xi(z)L_\eta(w) &\sim \frac{L_{[\xi,\eta]}(w)}{z-w} - \frac{1}{(z-w)^2} \left\{ c_1 \partial_\nu \xi^\mu(q(z)) \partial_\mu \eta^\nu(q(w)) \right. \\ &\quad \left. + c_2 \partial_\mu \xi^\mu(q(z)) \partial_\nu \eta^\nu(q(w)) \right\}, \\ T(z)L_\xi(w) &\sim \frac{d}{dw} \left(\frac{L_\xi(w)}{z-w} \right) + \frac{c_3}{(z-w)^3} \partial_\mu \xi^\mu(q(w)), \\ T(z)T(w) &\sim \frac{\dot{T}(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \frac{c_4/2}{(z-w)^4}, \\ J_X(z)J_Y(w) &\sim \frac{J_{[X,Y]}(w)}{z-w} + \frac{1}{(z-w)^2} \left\{ c_5 X^a(q(z)) Y^a(q(w)) \right. \\ &\quad \left. + c_8 \delta^a X^a(q(z)) \delta^b Y^b(q(w)) \right\}, \\ T(z)J_X(w) &\sim \frac{d}{dw} \left(\frac{J_X(w)}{z-w} \right) + \frac{c_6}{(z-w)^3} \delta^a X^a(q(w)), \\ L_\xi(z)J_X(w) &\sim \frac{J_{\xi X}(w)}{z-w} + \frac{c_7}{(z-w)^2} \partial_\mu \xi^\mu(q(z)) \delta^a X^a(q(w)), \\ L_\xi(z)q^\mu(w) &\sim \frac{1}{z-w} \xi^\mu(q(w)), \\ J_X(z)q^\mu(w) &\sim 0, \end{aligned} \quad (6.1)$$

$$\begin{aligned}
T(z)q^\mu(w) &\sim \frac{\dot{q}^\mu(w)}{z-w}, \\
q^\mu(z)q^\nu(w) &\sim 0.
\end{aligned}$$

Brackets:

$$\begin{aligned}
[\xi, \eta] &= \xi^\mu \partial_\mu \eta^\nu \partial_\nu - \eta^\nu \partial_\nu \xi^\mu \partial_\mu, \\
[X, Y] &= if^{abc} X^a Y^b J^c, \\
[f, g] &= f\dot{g} - g\dot{f}, \\
\xi X &= \xi^\mu \partial_\mu X + \partial_\mu \xi^\mu X.
\end{aligned} \tag{6.2}$$

Traces in the $\mathfrak{gl}(d) \oplus \mathfrak{g}$ representation $\varrho \oplus M$:

$$\begin{aligned}
\text{tr}_\varrho(I) &= \Delta_\varrho, \\
\text{tr}_\varrho(T_\nu^\mu) &= k_0(\varrho), \\
\text{tr}_\varrho(T_\rho^\mu T_\sigma^\nu) &= k_1(\varrho) \delta_\sigma^\mu \delta_\rho^\nu + k_2(\varrho) \delta_\rho^\mu \delta_\sigma^\nu, \\
\text{tr}_M(I) &= \Delta_M, \\
\text{tr}_M(M^a) &= z_M \delta^a, \\
\text{tr}_M(M^a M^b) &= y_M \delta^{ab} + w_M \delta^a \delta^b.
\end{aligned} \tag{6.3}$$

Note the additional term proportional to w_M added to the last equation. Such a term is possible in general, but vanishes if \mathfrak{g} is semisimple.

Abelian charges:

$$\begin{aligned}
c_1 &= 1 \pm \Delta_M \left\{ \binom{d+p+1}{d+2} \Delta_\varrho + \binom{d+p}{d} k_1(\varrho) \right\}, \\
c_2 &= \pm \Delta_M \left\{ \binom{d+p}{d+2} \Delta_\varrho + 2 \binom{d+p}{d+1} k_0(\varrho) + \binom{d+p}{d} k_2(\varrho) \right\}, \\
c_3 &= 1 \pm (2\lambda - 1) \Delta_M \left\{ \binom{d+p}{d+1} \Delta_\varrho + \binom{d+p}{d} k_0(\varrho) \right\} \\
c_4 &= 2d \pm 2(6\lambda^2 - 6\lambda + 1) \binom{d+p}{d} \Delta_\varrho \Delta_M, \\
c_5 &= \mp \binom{d+p}{d} y_M \Delta_\varrho, \\
c_6 &= \pm (2\lambda - 1) z_M \binom{d+p}{d} \Delta_\varrho. \\
c_7 &= \mp z_M \left\{ \binom{d+p}{d+1} \Delta_\varrho + \binom{d+p}{d} k_0(\varrho) \right\}, \\
c_8 &= \mp w_M \binom{d+p}{d} \Delta_\varrho.
\end{aligned} \tag{6.4}$$

Lie algebra generators:

$$\begin{aligned}
\mathcal{L}_\xi &= \frac{1}{2\pi i} \oint dz L_\xi(z) \\
\mathcal{J}_X &= \frac{1}{2\pi i} \oint dz J_X(z) \\
\mathcal{T}_f &= -\frac{1}{2\pi i} \oint dz f(z) T(z)
\end{aligned} \tag{6.5}$$

Lie brackets:

$$\begin{aligned}
[\mathcal{L}_\xi, \mathcal{L}_\eta] &= \mathcal{L}_{[\xi, \eta]} - \frac{1}{2\pi i} \oint dz \dot{q}^\rho(z) \left(c_1 \partial_\rho \partial_\nu \xi^\mu(q(z)) \partial_\mu \eta^\nu(q(z)) \right. \\
&\quad \left. + c_2 \partial_\rho \partial_\mu \xi^\mu(q(z)) \partial_\nu \eta^\nu(q(z)) \right), \\
[\mathcal{T}_f, \mathcal{L}_\xi] &= -\frac{c_3}{4\pi i} \oint dz \ddot{f}(z) \partial_\mu \xi^\mu(q(z)), \\
[\mathcal{T}_f, \mathcal{T}_g] &= \mathcal{T}_{[f, g]} - \frac{c_4}{24\pi i} \oint dz \ddot{f}(z) \dot{g}(z), \\
[\mathcal{J}_X, \mathcal{J}_Y] &= \mathcal{J}_{[X, Y]} + \frac{1}{2\pi i} \oint dz \dot{q}^\rho(z) \left(c_5 \partial_\rho X^a(q(z)) Y^a(q(z)) \right. \\
&\quad \left. + c_8 \delta^a \partial_\rho X^a(q(z)) \delta^b Y^b(q(z)) \right), \\
[\mathcal{T}_f, \mathcal{J}_X] &= -\frac{c_6}{4\pi i} \oint dz \ddot{f}(z) \delta^a X^a(q(z)), \\
[\mathcal{L}_\xi, \mathcal{J}_X] &= \mathcal{J}_{\xi X} + \frac{c_7}{2\pi i} \oint dz \dot{q}^\rho(z) \partial_\rho \partial_\mu \xi^\mu(q(z)) \delta^a X^a(q(z)), \\
[\mathcal{L}_\xi, q^\mu(z)] &= \xi^\mu(q(z)), \\
[\mathcal{J}_X, q^\mu(z)] &= 0, \\
[\mathcal{T}_f, q^\mu(z)] &= -f(z) \dot{q}^\mu(z), \\
[q^\mu(z), q^\nu(w)] &= 0.
\end{aligned} \tag{6.6}$$

The abelian charges agree with the result in [7], equations (5.6) and (6.5). The Lie brackets are the same as in equations (2.3) and (6.2) of that paper, apart from some cohomologically trivial terms, but all extensions have the opposite signs. The reason is that the standard OPE formalism used in the present article is based on highest-weight representations, whereas in [7] we instead considered representations of lowest-energy type. In physics the latter are more natural – energy is typically bounded from below rather

than from above – but the highest-weight convention is standard in CFT, and we wanted to facilitate comparison with the standard reference [4].

7 Conclusion

The technical results in this paper are identical to those in [7], but the presentation is quite different and hopefully more accessible. One improvement is that the Taylor coefficients have been hidden in generating functions, thus making the equations more similar to the analogous formulas for fields. That we really deal with p -jets is manifested by the replacement of delta functions by their p -jet counterparts, which can be multiplied with each other in a meaningful way.

The second improvement is the use of the standard OPE formalism to calculate brackets and extensions. The formalism in [7], albeit correct, was somewhat non-standard and cumbersome. The OPE formalism used here is simpler and more standard.

The obvious physical application of the multi-dimensional Virasoro algebra, in particular $Vir(4)$, is in quantum gravity. In fact, it is a necessary ingredient in any local quantum theory of gravity. Recall the standard argument why there can be no local observables in quantum gravity:

1. In quantum theory an observable is a gauge-invariant operator.
2. In general relativity, all spacetime diffeomorphisms are gauge symmetries.
3. Hence an observable in quantum gravity must commute with all diffeomorphisms, i.e. it can not depend on local coordinates.

The weak link in this argument is that in order to go from 2 to 3, one must assume that classical and quantum gravity have the same gauge symmetries. If the diffeomorphism algebra acquires an extension upon quantization, this assumption is false. $Vir(d)$ is the extension of $\mathfrak{vect}(d)$ which is necessary for local observables, just as a non-zero central charge is necessary for local observables in CFT. That a diffeomorphism algebra extension is necessary for locality was emphasized in [8].

8 Appendices

A Proof of the smearing properties (2.16) – (2.18)

Integrate by parts repeatedly.

$$\begin{aligned}
\int d^d y f(y) \delta_p(x, y) &= \sum_{|\mathbf{m}| \leq p} \frac{(-)^{|\mathbf{m}|}}{\mathbf{m}!} x^{\mathbf{m}} \int d^d y f(y) \partial_{\mathbf{m}} \delta(y) \\
&= \sum_{|\mathbf{m}| \leq p} \frac{1}{\mathbf{m}!} x^{\mathbf{m}} \int d^d y \partial_{\mathbf{m}} f(y) \delta(y) \\
&= \sum_{|\mathbf{m}| \leq p} \frac{1}{\mathbf{m}!} \partial_{\mathbf{m}} f(0) x^{\mathbf{m}} = f(x)|_p
\end{aligned}$$

$$\begin{aligned}
\int d^d y f(y) \partial_{\mu}^x \delta_p(x, y) &= \sum_{|\mathbf{m}| \leq p} \frac{(-)^{|\mathbf{m}|}}{\mathbf{m}!} m_{\mu} x^{\mathbf{m}-\mu} \int d^d y f(y) \partial_{\mathbf{m}} \delta(y) \\
&= \sum_{|\mathbf{m}| \leq p} \frac{1}{(\mathbf{m}-\mu)!} x^{\mathbf{m}} \int d^d y \partial_{\mathbf{m}} f(y) \delta(y) \\
&= \sum_{|\mathbf{m}| \leq p} \frac{1}{\mathbf{n}!} \partial_{\mathbf{n}+\mu} f(0) x^{\mathbf{n}} = \partial_{\mu} f(x)|_p
\end{aligned}$$

$$\begin{aligned}
\int d^d y f(y) \partial_{\mu}^y \delta_p(x, y) &= \sum_{|\mathbf{m}| \leq p} \frac{(-)^{|\mathbf{m}|}}{\mathbf{m}!} x^{\mathbf{m}} \int d^d y f(y) \partial_{\mathbf{m}+\mu} \delta(y) \\
&= - \sum_{|\mathbf{m}| \leq p} \frac{1}{\mathbf{m}!} x^{\mathbf{m}} \int d^d y \partial_{\mathbf{m}+\mu} f(y) \delta(y) \\
&= - \sum_{|\mathbf{m}| \leq p} \frac{1}{\mathbf{m}!} \partial_{\mathbf{m}+\mu} f(0) x^{\mathbf{m}} = -\partial_{\mu} f(x)|_p
\end{aligned}$$

B Some infinite sums

$$i. \quad A_{d,p} \equiv \sum_{|\mathbf{m}| \leq p} 1 = \binom{d+p}{d}, \quad (\text{B.1})$$

$$ii. \quad B_{d,p} \equiv \sum_{|\mathbf{m}| \leq p} m_{\mu} = \binom{d+p}{d+1}, \quad (\text{B.2})$$

$$iii. \quad C_{d,p} \equiv \sum_{|\mathbf{m}| \leq p} m_\mu^2 = \binom{d+p}{d+2} + \binom{d+p+1}{d+2}, \quad (B.3)$$

$$iv. \quad D_{d,p} \equiv \sum_{|\mathbf{m}| \leq p} m_\mu m_\nu = \binom{d+p}{d+2}, \quad \text{if } \mu \neq \nu, \quad (B.4)$$

$$v. \quad E_{d,p} \equiv \sum_{|\mathbf{m}| \leq p} m_\mu (m_\nu + 1) = \binom{d+p+1}{d+2}, \quad \text{if } \mu \neq \nu, \quad (B.5)$$

Proof: The relations are proven by induction, repeatedly using the recursion formula ([1], 2.1.1 II):

$$\begin{aligned} \binom{n+1}{m} &= \binom{n}{m} + \binom{n}{m-1} \\ &= \binom{n}{m} + \binom{n-1}{m-1} + \dots + \binom{n-m}{0} \\ &= \binom{n}{n-m} + \binom{n-1}{n-m} + \dots + \binom{n-m}{n-m}. \end{aligned} \quad (B.6)$$

i. Clearly, $A_{1,p} = \sum_{m=0}^p 1 = p+1$. For the recursive step,

$$\begin{aligned} A_{d,p} &= \sum_{i=0}^p \sum_{|\mathbf{m}| \leq p-i} 1 = \sum_{i=0}^p A_{d-1,p-i} \\ &= \sum_{i=0}^p \binom{d-1+p-i}{d-1} = \sum_{i=0}^p \binom{n-i}{n-p} \\ &= \binom{n+1}{p} = \binom{d+p}{p} = \binom{d+p}{d}, \end{aligned} \quad (B.7)$$

where \mathbf{m} is a multi-index with $d-1$ components, $n = d+p-1$, and we used the recursion formula.

ii. For $d=1$, $B_{1,p} = \sum_{m=0}^p m = \binom{p+1}{2}$. Let $i = m_{d-1}$ be the last component, and let \mathbf{m} denote a multi-index with $d-1$ components.

$$\begin{aligned} B_{d,p} &= \sum_{i=0}^p \sum_{|\mathbf{m}| \leq p-i} i = \sum_{i=0}^p i A_{d-1,p-i} \\ &= \sum_{j=0}^{p-1} (j+1) A_{d-1,p-j-1} \end{aligned} \quad (B.8)$$

$$= \sum_{j=0}^{p-1} j A_{d-1,p-j-1} + \sum_{j=0}^{p-1} A_{d-1,p-j-1}.$$

Thus

$$B_{d,p} = B_{d,p-1} + A_{d,p-1} = B_{d,p-1} + \binom{d+p-1}{d}. \quad (\text{B.9})$$

Using the recursion formula we verify that $B_{d,p} = \binom{d+p}{d+1}$.

iii. When $d = 1$,

$$C_{1,p} = \sum_{m=0}^p m^2 = \binom{p+1}{3} + \binom{p+2}{3} = \frac{1}{6}(p + 3p^2 + 2p^3), \quad (\text{B.10})$$

which is proven with a cubic ansatz for $C_{1,p}$ and identification of components.

The recursive step is

$$\begin{aligned} C_{d,p} &= \sum_{i=0}^p \sum_{|\mathbf{m}| \leq p-i} i^2 = \sum_{i=0}^p i^2 A_{d-1,p-i} \\ &= \sum_{j=0}^{p-1} (j^2 + 2j + 1) A_{d-1,p-j-1} \\ &= C_{d,p-1} + 2B_{d,p-1} + A_{d,p-1} = C_{d,p-1} + 2B_{d,p} - A_{d,p-1} \\ &= C_{d,p-1} + 2 \binom{d+p}{d+1} - \binom{d+p-1}{d} \\ &= C_{d,p-1} + \binom{d+p}{d+1} + \binom{d+p-1}{d+1}. \end{aligned} \quad (\text{B.11})$$

We use the recursion formula to verify that the given expression for $C_{d,p}$ indeed satisfies this relation.

iv. Let $i = m_{d-1}$, $j = m_{d-2}$, and let \mathbf{m} have $d-2$ components.

$$\begin{aligned} D_{d,p} &= \sum_{i=0}^p \sum_{j=0}^{p-i} \sum_{|\mathbf{m}| \leq p-i-j} ij = \sum_{i=0}^p \sum_{j=0}^{p-i} ij A_{d-2,p-i-j} \\ &= \sum_{i=0}^p i B_{d-1,p-i} = \sum_{k=0}^{p-1} (k+1) B_{d-1,p-k-1} \\ &= D_{d,p-1} + \sum_{k=0}^{p-1} \binom{d+p-2-k}{d} \\ &= D_{d,p-1} + \binom{d+p-1}{p-2} = D_{d,p-1} + \binom{d+p-1}{d+1}. \end{aligned} \quad (\text{B.12})$$

In the second last step, we made the substitution $n = d - 2 + p$, $n - m = d$, $m = p - 2$, and used the recursion formula to calculate the sum of binomial coefficients. We verify that $D_{d,p} = \binom{d+p}{d+2}$ satisfies this relation.

$v.$

$$\begin{aligned} E_{d,p} &= D_{d,p} + B_{d,p} \\ &= \binom{d+p}{d+2} + \binom{d+p}{d+1} = \binom{d+p+1}{d+2}. \end{aligned} \quad (\text{B.13})$$

C Evaluation of products of p -jet delta functions

Unlike the ordinary delta function $\delta(x)$, the p -jet delta functions can be multiplied with each other in a meaningful way. In this appendix we evaluate three expressions that are needed in the text.

Let $f(x)$ and $g(x)$ be some smearing functions, and denote by $f_0(x) = f(x) - f(0)$ and $g_0(x) = g(x) - g(0)$ the corresponding shifted functions. Clearly, $f_0(0) = g_0(0) = 0$. Since the functions are just shifted by a constant, the suffix 0 is not necessary in derivatives: $\partial_\mu f_0(x) = \partial_\mu f(x)$.

The following sums are needed in the text:

$$\begin{aligned} i. \quad & \iint d^d x d^d y f(x) g(y) \delta_p(x, y) \delta_p(y, x) \\ &= \binom{d+p}{d} f(0) g(0), \end{aligned} \quad (\text{C.1})$$

$$\begin{aligned} ii. \quad & \iint d^d x d^d y f_0(x) g(y) \partial_\mu^x \delta_p(x, y) \delta_p(y, x) \\ &= \binom{d+p}{d+1} \partial_\mu f(0) g(0), \end{aligned} \quad (\text{C.2})$$

$$\begin{aligned} iii. \quad & \iint d^d x d^d y f_0(x) g_0(y) \partial_\mu^x \delta_p(x, y) \partial_\nu^y \delta_p(y, x) \\ &= \binom{d+p+1}{d+2} \partial_\nu f(0) \partial_\mu g(0) + \binom{d+p}{d+2} \partial_\mu f(0) \partial_\nu g(0). \end{aligned} \quad (\text{C.3})$$

Proof:

$i.$ By definition of the p -jet delta function, the LHS reads

$$\begin{aligned} & \iint d^d x d^d y f(x) g(y) \sum_m \frac{(-)^{|\mathbf{m}|}}{\mathbf{m}!} x^{\mathbf{m}} \partial_{\mathbf{m}} \delta(y) \sum_n \frac{(-)^{|\mathbf{n}|}}{\mathbf{n}!} y^{\mathbf{n}} \partial_{\mathbf{n}} \delta(x) \\ &= \sum_{\mathbf{m}, \mathbf{n}} \frac{1}{\mathbf{m}! \mathbf{n}!} \iint d^d x d^d y \partial_{\mathbf{n}} (x^{\mathbf{m}} f(x)) \partial_{\mathbf{m}} (y^{\mathbf{n}} g(y)) \delta(x) \delta(y), \end{aligned} \quad (\text{C.4})$$

where we integrated by parts \mathbf{m} times in x and \mathbf{n} times in y . Now,

$$\int d^d x \partial_{\mathbf{n}}(x^{\mathbf{m}} f(x)) \delta(x) = 0, \quad (\text{C.5})$$

unless $\mathbf{n} \geq \mathbf{m}$ (which means that $n_\mu > m_\mu$ for all μ), because otherwise the integrand would be proportional to x^μ , and the delta function would kill it. At the same time, we must also have $\mathbf{m} \geq \mathbf{n}$, because otherwise the integration over y would vanish. Hence $\mathbf{m} = \mathbf{n}$ and (C.4) is equal to

$$\begin{aligned} & \sum_{\mathbf{m}} \frac{1}{(\mathbf{m}!)^2} \iint d^d x d^d y \partial_{\mathbf{m}}(x^{\mathbf{m}} f(x)) \partial_{\mathbf{m}}(y^{\mathbf{m}} g(y)) \delta(x) \delta(y) \\ &= \sum_{\mathbf{m}} \frac{1}{(\mathbf{m}!)^2} \iint d^d x d^d y (\mathbf{m}! f(x)) (\mathbf{m}! g(y)) \delta(x) \delta(y) \\ &= \sum_{\mathbf{m}} 1 \cdot f(0) g(0) \\ &= A_{d,p} f(0) g(0), \end{aligned} \quad (\text{C.6})$$

where $A_{d,p}$ is defined in (B.1).

ii. The LHS becomes

$$\begin{aligned} & \iint d^d x d^d y f_0(x) g(y) \sum_{\mathbf{m}} \frac{(-)^{|\mathbf{m}|}}{\mathbf{m}!} m_\mu x^{\mathbf{m}-\mu} \partial_{\mathbf{m}} \delta(y) \sum_{\mathbf{n}} \frac{(-)^{|\mathbf{n}|}}{\mathbf{n}!} y^{\mathbf{n}} \partial_{\mathbf{n}} \delta(x) \\ &= \sum_{\mathbf{m}, \mathbf{n}} \frac{1}{(\mathbf{m}-\mu)! \mathbf{n}!} \iint d^d x d^d y \partial_{\mathbf{n}}(x^{\mathbf{m}-\mu} f_0(x)) \partial_{\mathbf{m}}(y^{\mathbf{n}} g(y)) \delta(x) \delta(y), \end{aligned} \quad (\text{C.7})$$

after repeated integration by parts. In order to kill off all factors of x^μ , we must have $\mathbf{n} \geq \mathbf{m} - \mu$. There are two ways this can be achieved:

- a) $\mathbf{m} = \mathbf{n}$,
- b) $\mathbf{n} = \mathbf{m} - \mu$, $\mathbf{m} = \mathbf{n} + \mu$.

However, in the second case the expression is proportional to

$$\int d^d x \partial_{\mathbf{n}}(x^{\mathbf{n}} f_0(x)) \delta(x) = \mathbf{n}! f_0(0) = 0, \quad (\text{C.8})$$

because the smearing function $f_0(x)$ is shifted to make $f_0(0) = 0$. Hence only the first case contributes to the result, which becomes

$$\begin{aligned} & \sum_{\mathbf{m}} \frac{1}{(\mathbf{m}-\mu)! \mathbf{m}!} \iint d^d x d^d y m_\mu (\mathbf{m}-\mu)! \partial_\mu f_0(x) \mathbf{m}! g(y) \delta(x) \delta(y) \\ &= \sum_{\mathbf{m}} m_\mu \cdot \partial_\mu f_0(0) g(0) \\ &= B_{d,p} \partial_\mu f(0) g(0), \end{aligned} \quad (\text{C.9})$$

where we used (B.2) and $\partial_\mu f_0(0) = \partial_\mu f(0)$ in the last step.

iii. The LHS becomes

$$\begin{aligned}
& \iint d^d x d^d y f_0(x) g_0(y) \sum_{\mathbf{m}} \frac{(-)^{|\mathbf{m}|}}{\mathbf{m}!} m_\mu x^{\mathbf{m}-\mu} \partial_{\mathbf{m}} \delta(y) \times \\
& \quad \times \sum_{\mathbf{n}} \frac{(-)^{|\mathbf{n}|}}{\mathbf{n}!} n_\nu y^{\mathbf{n}-\nu} \partial_{\mathbf{n}} \delta(x) \\
& = \sum_{\mathbf{m}, \mathbf{n}} \frac{1}{(\mathbf{m}-\mu)! (\mathbf{n}-\nu)!} \times \\
& \quad \times \iint d^d x d^d y \partial_{\mathbf{n}} (x^{\mathbf{m}-\mu} f_0(x)) \partial_{\mathbf{m}} (y^{\mathbf{n}-\nu} g_0(y)) \delta(x) \delta(y).
\end{aligned} \tag{C.10}$$

There are now five cases that may survive:

- a) $\mathbf{m} = \mathbf{n}$,
- b) $\mathbf{n} = \mathbf{m} - \mu$, $\mathbf{m} = \mathbf{n} + \mu$,
- c) $\mathbf{m} = \mathbf{n} - \nu$, $\mathbf{n} = \mathbf{m} + \nu$,
- d) $\mu \neq \nu$ and $\mathbf{n} = \mathbf{m} - \mu + \nu$, $\mathbf{m} = \mathbf{n} + \mu - \nu$,
- e) $\mu = \nu$ and $\mathbf{n} = \mathbf{m} - 2\mu$, $\mathbf{m} = \mathbf{n} + 2\mu$.

In case a), (C.10) equals

$$\begin{aligned}
& \sum_{\mathbf{m}} \frac{1}{(\mathbf{m}-\mu)! (\mathbf{m}-\nu)!} m_\mu (\mathbf{m}-\mu)! \partial_\mu f_0(0) m_\nu (\mathbf{m}-\nu)! \partial_\nu g_0(0) \\
& = \sum_{\mathbf{m}} m_\mu m_\nu \cdot \partial_\mu f_0(0) \partial_\nu g_0(0) \\
& = \begin{cases} C_{d,p} \partial_\mu f(0) \partial_\nu g(0), & \text{if } \mu = \nu, \\ D_{d,p} \partial_\mu f(0) \partial_\nu g(0), & \text{if } \mu \neq \nu, \end{cases}
\end{aligned} \tag{C.11}$$

where the sum was evaluated in (B.3) and (B.4).

Cases b) and c) both vanish, because the expressions are proportional to

$$\begin{aligned}
& \int d^d x \partial_{\mathbf{n}} (x^{\mathbf{n}} f_0(x)) \delta(x) = \mathbf{n}! f_0(0) = 0, \\
& \int d^d y \partial_{\mathbf{m}} (y^{\mathbf{m}} g_0(y)) \delta(y) = \mathbf{m}! g_0(0) = 0,
\end{aligned} \tag{C.12}$$

respectively.

Case d) is already covered by case a) if $\mu = \nu$. If $\mu \neq \nu$, equation (C.10) reads

$$\begin{aligned}
& \sum_{\mathbf{m}} \frac{1}{(\mathbf{m} - \mu)!(\mathbf{n} - \nu)!} \iint d^d x d^d y \partial_{\mathbf{n}}(x^{\mathbf{n}-\nu} f_0(x)) \partial_{\mathbf{m}}(y^{\mathbf{m}-\mu} g_0(y)) \delta(x) \delta(y) \\
&= \sum_{\mathbf{m}} n_{\nu} \partial_{\nu} f_0(0) m_{\mu} \partial_{\mu} g_0(0) \\
&= \sum_{\mathbf{m}} m_{\mu} (\mathbf{m} - \mu + \nu)_{\nu} \partial_{\nu} f(0) \partial_{\mu} g(0).
\end{aligned} \tag{C.13}$$

We now note that $\nu_{\nu} = 1$ and $\mu_{\nu} = 0$ because $\mu \neq \nu$. Using (B.5) we arrive at

$$\sum_{\mathbf{m}} m_{\mu} (m_{\nu} + 1) \partial_{\nu} f(0) \partial_{\mu} g(0) = E_{d,p} \partial_{\nu} f(0) \partial_{\mu} g(0). \tag{C.14}$$

Finally we have case e), but this vanishes because it is proportional to

$$\int d^d x \partial_{\mathbf{n}}(x^{\mathbf{n}+2\mu} f_0(x)) \delta(x) = 0, \tag{C.15}$$

and there are not enough derivatives to kill all powers of x .

Summing up the non-zero contributions from cases a) and d), the result for the integral in (C.10) is

$$C_{d,p} \partial_{\mu} f(0) \partial_{\nu} g(0) \tag{C.16}$$

if $\mu = \nu$, and

$$E_{d,p} \partial_{\nu} f(0) \partial_{\mu} g(0) + D_{d,p} \partial_{\mu} f(0) \partial_{\nu} g(0) \tag{C.17}$$

if $\mu \neq \nu$. However, since $C_{d,p} = E_{d,p} + D_{d,p}$, the latter expression is equal to (C.16) when $\mu = \nu$, so the covariant result (C.17) holds irrespective of whether μ equals ν or not.

D Evaluation of the $Vir(d)$ OPE (4.20)

In this appendix we evaluate the OPE between the three partial $Vir(d)$ generators listed in (4.17). Since none of the generators involves any derivatives of z , the Wick contractions must be of the form

$$L_{\xi}^i(z) L_{\eta}^j(w) \sim \frac{R_{\xi,\eta}^{ij}(w)}{(z-w)} + \frac{Z_{\xi,\eta}^{ij}(w)}{(z-w)^2}. \tag{D.1}$$

To reduce writing, we suppress arguments where so can be done without obscuring the meaning.

First consider the regular terms.

$$\begin{aligned}
R_{\xi,\eta}^{00} &= : \xi^\mu(q) p_\mu :: \eta^\nu(q) p_\nu : \\
&\sim \eta^\nu(q) \partial_\nu \xi^\mu(q) p_\mu - \xi^\mu(q) \partial_\mu \eta^\nu(q) p_\mu \\
&= L_{[\xi,\eta]}^0
\end{aligned} \tag{D.2}$$

$$\begin{aligned}
R_{\xi,\eta}^{01} &= - : \xi^\mu(q) p_\mu : \int (\eta^\nu(x+q) - \eta^\nu(q)) : \pi \partial_\nu \phi : \\
&\sim \xi^\mu(q) \int (\partial_\mu \eta^\nu(x+q) - \partial_\mu \eta^\nu(q)) : \pi \partial_\nu \phi :
\end{aligned} \tag{D.3}$$

$$\begin{aligned}
R_{\xi,\eta}^{10} &= - \int (\xi^\mu(x+q) - \xi^\mu(q)) : \pi \partial_\mu \phi :: \eta^\nu(q) p_\nu : \\
&\sim - \eta^\nu(q) \int (\partial_\nu \xi^\mu(x+q) - \partial_\nu \xi^\mu(q)) : \pi \partial_\mu \phi :
\end{aligned} \tag{D.4}$$

To evaluate $R_{\xi,\eta}^{11}$, define

$$\xi_0^\mu(x) = \xi^\mu(x+q) - \xi^\mu(q) \tag{D.5}$$

We then have

$$\begin{aligned}
R_{\xi,\eta}^{11} &= \iint \xi_0^\mu(x) \eta_0^\nu(y) : \pi(x) \partial_\mu \phi(x) :: \pi(y) \partial_\nu \phi(y) : \\
&\sim \iint \xi_0^\mu(x) \eta_0^\nu(y) \left(\partial_\mu^x \delta_p(x, y) : \pi(x) \partial_\nu \phi(y) : \right. \\
&\quad \left. - \partial_\nu^y \delta_p(y, x) : \pi(y) \partial_\mu \phi(x) : \right) \\
&= \int \xi_0^\mu \partial_\mu \eta^\nu : \pi \partial_\nu \phi : + \int \xi_0^\mu \eta_0^\nu : \pi \partial_\mu \partial_\nu \phi : \\
&\quad - \int \eta_0^\nu \partial_\nu \xi^\mu : \pi \partial_\mu \phi : - \int \xi_0^\mu \eta_0^\nu : \pi \partial_\nu \partial_\mu \phi : \\
&= \int (\xi^\mu(x+q) - \xi^\mu(q)) \partial_\mu \eta^\nu(x+q) : \pi \partial_\nu \phi : \\
&\quad - \int (\eta^\nu(x+q) - \eta^\nu(q)) \partial_\nu \xi^\mu(x+q) : \pi \partial_\mu \phi : .
\end{aligned} \tag{D.6}$$

Partial integration in the second step was performed using (2.17). The sum of (D.3), (D.4) and (D.6) is

$$\begin{aligned}
R_{\xi,\eta}^{01} + R_{\xi,\eta}^{10} + R_{\xi,\eta}^{11} &= \int (\xi^\mu \partial_\mu \eta^\nu)_0 : \pi \partial_\nu \phi : - \int (\eta^\nu \partial_\nu \xi^\mu)_0 : \pi \partial_\mu \phi : \\
&= L_{[\xi,\eta]}^1.
\end{aligned} \tag{D.7}$$

The next two terms are

$$\begin{aligned}
R_{\xi,\eta}^{02} &= - : \xi^\mu(q) p_\mu : \int \partial_\rho \eta^\nu(x+q) : \pi T_\nu^\rho \phi : \\
&\sim \xi^\mu(q) \int \partial_\mu \partial_\rho \eta^\nu(x+q) : \pi T_\nu^\rho \phi :
\end{aligned} \tag{D.8}$$

$$\begin{aligned}
R_{\xi,\eta}^{12} &= \iint \xi_0^\mu(x) \partial_\rho \eta^\nu(y) : \pi(x) \partial_\mu \phi(x) : : \pi(y) T_\nu^\rho \phi(y) : \\
&\sim \iint \xi_0^\mu(x) \partial_\rho \eta^\nu(y) \left(\partial_\mu^x \delta_p(x,y) : \pi(x) T_\nu^\rho \phi(y) : \right. \\
&\quad \left. - \delta_p(y,x) : \pi(y) T_\nu^\rho \partial_\mu \phi(x) : \right) \\
&= \int \left(\xi_0^\mu \partial_\mu \partial_\rho \eta^\nu : \pi T_\nu^\rho \phi : + \xi_0^\mu \partial_\rho \eta^\nu(y) : \pi T_\nu^\rho \partial_\mu \phi : \right. \\
&\quad \left. - \xi_0^\mu \partial_\rho \eta^\nu(y) : \pi T_\nu^\rho \partial_\mu \phi : \right) \\
&= \int (\xi^\mu(x+q) - \xi^\mu(q)) \partial_\mu \partial_\rho \eta^\nu : \pi T_\nu^\rho \phi :
\end{aligned} \tag{D.9}$$

Summing the last two contribution, we find

$$R_{\xi,\eta}^{02} + R_{\xi,\eta}^{12} = \int \xi^\mu(x+q) \partial_\mu \partial_\rho \eta^\nu(x+q) : \pi T_\nu^\rho \phi : \tag{D.10}$$

and analogously

$$R_{\xi,\eta}^{20} + R_{\xi,\eta}^{21} = - \int \eta^\nu(x+q) \partial_\nu \partial_\rho \xi^\mu(x+q) : \pi T_\mu^\rho \phi : \tag{D.11}$$

Finally,

$$\begin{aligned}
R_{\xi,\eta}^{22} &= \iint \partial_\rho \xi^\mu \partial_\sigma \eta^\nu : \pi T_\mu^\rho \phi : : \pi T_\nu^\sigma \phi : \\
&\sim \iint \partial_\rho \xi^\mu \partial_\sigma \eta^\nu : \pi [T_\mu^\rho, T_\nu^\sigma] \phi : \\
&= \int \partial_\rho \xi^\mu \partial_\mu \eta^\nu : \pi T_\nu^\rho \phi : - \int \partial_\rho \eta^\nu \partial_\nu \xi^\mu : \pi T_\mu^\rho \phi :
\end{aligned} \tag{D.12}$$

Summing the last few contribution, we find that the sum $R_{\xi,\eta}^{02} + R_{\xi,\eta}^{12} +$

$R_{\xi,\eta}^{20} + R_{\xi,\eta}^{21} + R_{\xi,\eta}^{22}$ equals

$$\begin{aligned}
& \int (\partial_\rho \xi^\mu \partial_\mu \eta^\nu + \xi^\mu \partial_\mu \partial_\rho \eta^\nu) : \pi T_\nu^\rho \phi : \\
& - \int (\partial_\rho \eta^\nu \partial_\nu \xi^\mu + \eta^\nu \partial_\nu \partial_\rho \xi^\mu) : \pi T_\mu^\rho \phi : \\
& = \int \partial_\rho [\xi, \eta]^\mu : \pi T_\mu^\rho \phi : = L_{[\xi, \eta]}^2
\end{aligned} \tag{D.13}$$

Using (D.2), (D.7) and (D.13), we finally have

$$\sum_{i=0}^2 \sum_{j=0}^2 R_{\xi,\eta}^{ij} = L_{[\xi, \eta]}^0 + L_{[\xi, \eta]}^1 + L_{[\xi, \eta]}^2 = L_{[\xi, \eta]}. \tag{D.14}$$

We now turn to the double Wick contractions.

$$Z_{\xi,\eta}^{00} = : \xi^\mu(q) \underbrace{p_\mu :: \eta^\nu(q) p_\nu} : = -\partial_\nu \xi^\mu(q) \partial_\mu \eta^\nu(q) \tag{D.15}$$

The other terms involving the observer's momentum vanish,

$$Z_{\xi,\eta}^{01}(z, w) = Z_{\xi,\eta}^{10}(z, w) = Z_{\xi,\eta}^{02}(z, w) = Z_{\xi,\eta}^{20}(z, w) = 0, \tag{D.16}$$

because in these terms only a single Wick contraction is possible.

$$\begin{aligned}
Z_{\xi,\eta}^{11} &= \iint \xi_0^\mu(x) \eta_0^\nu(y) : \underbrace{\pi(x) \partial_\mu \phi(x) :: \pi(y) \partial_\nu \phi(y)} : \\
&= \mp \text{tr } I \iint \xi_0^\mu(x) \eta_0^\nu(y) \partial_\mu^x \delta_p(x, y) \partial_\nu^y \delta_p(y, x) \\
&= \mp \Delta_\varrho \left(E_{d,p} \partial_\nu \xi^\mu(q) \partial_\mu \eta^\nu(q) + D_{d,p} \partial_\mu \xi^\mu(q) \partial_\nu \eta^\nu(q) \right),
\end{aligned} \tag{D.17}$$

where we used (C.3) and the fact that there are $\Delta_\varrho = \text{tr } I$ different fields that contribute to the sum.

$$\begin{aligned}
Z_{\xi,\eta}^{12} &= \iint \xi_0^\mu(x) \partial_\rho \eta^\nu(y) : \underbrace{\pi(x) \partial_\mu \phi(x) :: \pi(y) T_\nu^\rho \phi(y)} : \\
&= \mp \text{tr } T_\nu^\rho \iint \xi_0^\mu(x) \partial_\rho \eta^\nu(y) \partial_\mu^x \delta_p(x, y) \delta_p(y, x) \\
&= \mp B_{d,p} k_0(\varrho) \partial_\mu \xi^\mu(q) \partial_\nu \eta^\nu(q)
\end{aligned} \tag{D.18}$$

Here we used (C.2) to evaluate the sum and that the trace of the $\mathfrak{gl}(d)$ generators $\text{tr } T_\nu^\mu = k_0(\varrho) \delta_\nu^\mu$. By symmetry we immediately obtain $Z_{\xi,\eta}^{21} =$

$Z_{\xi,\eta}^{12}$.

$$\begin{aligned}
Z_{\xi,\eta}^{22} &= \iint \partial_\rho \xi^\mu(x) \partial_\sigma \eta^\nu(y) : \underbrace{\pi(x) T_\mu^\rho \phi(x) :: \pi(y) T_\nu^\sigma \phi(y)} : \\
&= \mp \text{tr} T_\mu^\rho T_\nu^\sigma \iint \partial_\rho \xi^\mu(x) \partial_\sigma \eta^\nu(y) \delta_p(x, y) \delta_p(y, x) \\
&= \mp A_{d,p} (k_1(\varrho) \delta_\nu^\rho \delta_\mu^\sigma + k_2(\varrho) \delta_\mu^\rho \delta_\nu^\sigma) \partial_\rho \xi^\mu(q) \partial_\sigma \eta^\nu(q),
\end{aligned} \tag{D.19}$$

where we used (C.1) and (4.22).

Summing the nonzero contributions,

$$Z_{\xi,\eta} = Z_{\xi,\eta}^{00} + Z_{\xi,\eta}^{11} + 2Z_{\xi,\eta}^{12} + Z_{\xi,\eta}^{22}, \tag{D.20}$$

we see that the extension is of the form (4.21) and the abelian charges are given by (4.23).

E Proof of the results in subsection 4.4

The OPE will be of the form

$$L_\xi(z) J_X(w) \sim \frac{H_{\xi,X}(w)}{z-w} + \frac{W_{\xi,X}(z, w)}{(z-w)^2}. \tag{E.1}$$

First we calculate the regular term:

$$\begin{aligned}
H_{\xi,X}(w) &= \xi^\mu(q) \int \partial_\mu X^a : \pi M^a \phi : + \int X^a : (-\xi_0^\mu \partial_\mu \pi + \partial_\nu \xi^\mu \pi T_\mu^\nu) M^a \phi : \\
&\quad + \int X^a : \pi M^a (-\xi_0^\mu \partial_\mu \phi - \partial_\nu \xi^\mu T_\mu^\nu) \phi : \\
&= \xi^\mu(q) \int \left(\partial_\mu X^a : \pi M^a \phi : + X^a \partial_\mu (: \pi M^a \phi :) \right) \\
&\quad - \int X^a \xi^\mu \partial_\mu (: \pi M^a \phi :).
\end{aligned} \tag{E.2}$$

The first term is a total derivative, and the second becomes after an integration by parts:

$$H_{\xi,X}(w) = \int \partial_\mu (\xi^\mu(x+q) X^a(x+q)) : \pi M^a \phi :. \tag{E.3}$$

We recognize the transformation law for a density of weight one.

The extension only involves contributions from $L_\xi^1(z)$ and $L_\xi^2(z)$, because there is no double contraction between $L_\xi^0(z)$ and $J_X(w)$.

$$\begin{aligned}
W_{\xi,X} &= \iint \left(\xi_0^\mu(x) X^a(y) : \underbrace{\pi(x) \partial_\mu \phi(x) :: \pi(y) M^a \phi(y)} : \right. \\
&\quad \left. + \partial_\nu \xi^\mu(x) X^a(y) : \underbrace{\pi(x) T_\mu^\nu \phi(x) :: \pi(y) M^a \phi(y)} : \right) \\
&= \mp \text{tr } M^a \iint \left(\xi_0^\mu X^a \partial_\mu^x \delta_p(x, y) \delta_p(y, x) \text{tr } I \right. \\
&\quad \left. + \partial_\nu \xi^\mu X^a \delta_p(x, y) \delta_p(y, x) \text{tr } T_\nu^\mu \right) \\
&= \mp z_M \delta^a \left(B_{d,p} \Delta_\varrho \partial_\mu \xi^\mu(q) X^a(q) + A_{d,p} k_0(\varrho) \delta_\mu^\nu \partial_\nu \xi^\mu(q) X^a(q) \right) \\
&= \mp z_M \left(B_{d,p} \Delta_\varrho + A_{d,p} k_0(\varrho) \right) \partial_\mu \xi^\mu(q) \delta^a X^a(q),
\end{aligned} \tag{E.4}$$

where we used (C.1) and (C.2) to evaluate the products of delta functions. The form of the OPE and the expression for the abelian charge c_7 (4.35) – (4.36) are thus confirmed.

F Proof of the equations in section 5

Since the reparametrization generators depend on z -derivatives, it is useful to list the OPEs between z -derivatives of the fundamental fields:

$$\begin{aligned}
\phi(x, z) \pi(y, w) &\sim \frac{1}{z-w} \delta_p(x, y) & \pi(x, z) \phi(y, w) &\sim \frac{\mp 1}{z-w} \delta_p(y, x) \\
\dot{\phi}(x, z) \pi(y, w) &\sim \frac{-1}{(z-w)^2} \delta_p(x, y) & \pi(x, z) \dot{\phi}(y, w) &\sim \frac{\mp 1}{(z-w)^2} \delta_p(y, x) \\
\phi(x, z) \dot{\pi}(y, w) &\sim \frac{1}{(z-w)^2} \delta_p(x, y) & \dot{\pi}(x, z) \phi(y, w) &\sim \frac{\pm 1}{(z-w)^2} \delta_p(y, x) \\
\dot{\phi}(x, z) \dot{\pi}(y, w) &\sim \frac{-2}{(z-w)^3} \delta_p(x, y) & \dot{\pi}(x, z) \dot{\phi}(y, w) &\sim \frac{\pm 2}{(z-w)^3} \delta_p(y, x)
\end{aligned}$$

In general, the $T(z)T(w)$ extension is obtained by double contractions of four terms. For brevity we only consider the case $\lambda = 0$ where the extension

only consists of a single term:

$$\begin{aligned}
T(z)T(w) &\sim \iint : \underbrace{\pi(x, z)\dot{\phi}(x, z) :: \pi(y, w)\dot{\phi}(y, w)} : \\
&= \text{tr } I \iint \frac{-1}{(z-w)^2} \delta_p(x, y) \frac{\mp 1}{(z-w)^2} \delta_p(y, x) \quad (\text{F.1}) \\
&= \frac{\pm 1}{(z-w)^4} \Delta_\varrho \Delta_M A_{d,p}.
\end{aligned}$$

The results for non-zero λ is obtained analogously. The central charge c_4 simply equals the usual Virasoro central charge (5.5) times the number of conformal fields $\binom{d+p}{d} \Delta_\varrho \Delta_M$.

The extension in the $T(z)J_X(w)$ OPE is

$$\begin{aligned}
&\int ((\lambda - 1) : \pi \dot{\phi} : + \lambda : \dot{\pi} \phi :) \int : \pi X \phi : \\
&\sim \iint \text{tr } X \frac{\pm(2\lambda - 1)}{(z-w)^3} \text{tr } I \delta_p(x, y) \delta_p(y, x) \quad (\text{F.2}) \\
&= \pm \text{tr } X(q) \frac{2\lambda - 1}{(z-w)^3} \Delta_\varrho A_{d,p} \\
&= \pm z_M \delta^a X^a(q) \frac{2\lambda - 1}{(z-w)^3} \Delta_\varrho A_{d,p},
\end{aligned}$$

which is (5.12).

The extension of the $T(z)L_\xi(w)$ OPE is

$$\begin{aligned}
&\int ((\lambda - 1) : \pi \dot{\phi} : + \lambda : \dot{\pi} \phi :) \int (\xi_0^\mu : \pi \partial_\mu \phi : + \partial_\nu \xi^\mu : \pi T_\mu^\nu \phi : \\
&\sim \frac{\pm(2\lambda - 1)}{(z-w)^3} \iint \left(\xi_0^\mu \text{tr } I \delta_p(x, y) \partial_\mu^y \delta_p(y, x) + \partial_\nu \xi^\mu \text{tr } T_\mu^\nu \delta_p(x, y) \delta_p(y, x) \right) \\
&= \pm \frac{2\lambda - 1}{(z-w)^3} \left(\partial_\mu \xi^\mu(q) \Delta_\varrho B_{d,p} + \partial_\mu \xi^\mu(q) k_0(\varrho) A_{d,p} \right) \Delta_M, \quad (\text{F.3})
\end{aligned}$$

which is (5.11).

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